Higher dimensional Charged Black Holes

as constrained systems

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Abstract

We construct a Lagrangian and Hamiltonian formulation for charged black holes in a d-dimensional maximally symmetric spherical space. By considering first new variables that give raise to an interesting dimensional reduction of the problem, we show that the introduction of a charge term is compatible with classical solutions to Einstein equations. In fact, we derive the well-known solutions for charged black holes, specially in the case of d=4, where the Reissner-Nordström solution holds, without reference to Einstein field equations. We argue that our procedure may be of help for clarifying symmetries and dynamics of black holes, as well as some quantum aspects.

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1. Introduction

The static solutions of Einstein field equations associated with spherical symmetric spaces have been extensively studied in the literature. However, in the past decades, mainly due to the emergence of string theory, there has been a renewed interest in studying black hole solutions involving extra dimensions [1]-[7]. In fact, the study of black holes in several dimensions is a strong indicative of the level of consistency developed in different approaches for solving the problem of quantum gravity [8]-[12].

In this work, we focus our attention in a method for obtaining the static solutions for charged black holes, that permit us to make the analysis in complete analogy to that of a relativistic point particle. In particular, we find that charged black holes can be treated as a constrainned hamiltonian system.

It is well known that, by taking the radial coordinate as an evolution parameter, one can describe the black hole dynamics in terms of a type of Hamiltonian. Recently, however, such Hamiltonian structure was described in such a way that it establishes a clear connection with constrained Hamiltonian systems (see [3] and references therein). In this work we start with a very general form for the metric in order to generalize such constrained Hamiltonian system to higher dimensions.

Moreover, starting with the Einstein-Maxwell action, we prove that our approach can be extended to the case of charged black holes. In this case we verify that one can recover the Reissner-Nordström solution. One of the advantages of our procedure is that one can perform a similar analysis to the relativistic point particle. For instance, the invariance of the corresponding action under an arbitrary change of the radial parameter can be linked to a Lagrange multiplier. In this way, fixing this Lagrange multiplier is equivalent to set a gauge in order to recover the Reissner-Nordström solution. Moreover, considering new variables the Lagrangian formulation is related to a Hamiltonian formulation that allows to treat the problem of charged black holes as a constrained system. We argue that this procedure may help to tackle the problem of quantum black holes [13][14].

The structure of this paper is the following: In section 2, we briefly review the general method of obtaining the Reissner-Nordström solution from the Maxwell-Einstein field equations. In section 3, we introduce the Lagrangian formulation in terms of a particular set of variables that simplify the analysis. All the developments made until this point lead us, in section 4, to introduce an associated Hamiltonian which allows to take a view of the charged black holes in higher dimensions as a constrained Hamiltonian system. We conclude with various remarks, in section 5, summarizing the work and mentioning some

perspectives for future explorations on the subject. In particular, we remark that our analysis can be useful for quantum black holes in higher dimensions.

2. The Einstein-Maxwell model for charged black holes

We start by considering the Einstein-Maxwell action in d dimensions, given by [4]

$$S = \frac{1}{16\pi G_d} \int_{M^d} \sqrt{-\gamma} \left(R - F_{\mu\nu}F^{\mu\nu} \right), \tag{1}$$

where G_d is the gravitational constant in d dimensions, R is the scalar curvature. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor defined in terms of a potential A_μ, and γ is the determinant of the metric $\gamma_{\mu\nu}$. To guarantee static spherical symmetry, we assume the general metric

$$ds^2 = -e^{f(r)}dt^2 + e^{h(r)}dr^2 + \varphi^2(r)\tilde{\gamma}_{ij}(\xi^k)d\xi^i d\xi^j. \tag{2}$$

Here, we have specified the functional form for f, h and φ in terms of r and considered the speed of light $c = 1$. The submetric $\tilde{\gamma}_{ij}(\xi^k)$ corresponds to a maximally symmetric subspace in $(d-2)$-dimensions with curvilinear coordinates ξ^i that are independent of time t and r. The notation in this article is as follows: greek indices like μ, ν run from 0 to $d-1$, while latin indices like i, j run from 2 to $d-1$; also, as a convenient notation we will be referring to derivatives respect to r with dot notation, that is $\dot{f} \equiv \frac{df}{dr}$ and $\ddot{f} \equiv \frac{d^2 f}{dr^2}$, for instance.

Here, we shall consider electrically charged black holes, with the only non-vanishing strength field tensor component $F_{10} = -F_{01} = \partial_1 A_0 = \dot{\chi}$, where χ is the electric potential.

Taking into account the form of the metric (2), we have the relation between determinants $\sqrt{-\gamma} = e^{\frac{f+h}{2}}\varphi^{(d-2)}\sqrt{\tilde{\gamma}}$ and also $F_{\lambda\tau}F^{\lambda\tau} = -2e^{-(h+f)}\dot{\chi}^2$. In this manner, by using the curvature scalar R corresponding to the metric (2) (cf. Appendix), we find that the action (1), up to a total derivative, takes the form

$$S = \frac{(d-2)}{16\pi G_d} \int_{M^d} \sqrt{\tilde{\gamma}} \Big\{ \varphi^{(d-2)} e^{\frac{f-h}{2}} \left[(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + \dot{f}\frac{\dot{\varphi}}{\varphi} \right]$$

$$+ k(d-3)e^{\frac{f+h}{2}}\varphi^{(d-4)} + \frac{2}{(d-2)}e^{\frac{-(f+h)}{2}}\varphi^{(d-2)}\dot{\chi}^2 \Big\}. \tag{3}$$

We shall focus our attention on the case $d \geq 4$. If we further define $\mathcal{F} \equiv e^{\frac{f}{2}}$

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and $\Omega \equiv e^{\frac{h}{2}}$, then we can express (3) as

$$S = \tfrac{(d-2)}{16\pi G_d} \int_{M^d} \sqrt{\tilde{\gamma}} \Big\{ \mathcal{F}\Omega^{-1}\varphi^{(d-4)} \Big[(d-3)\dot{\varphi}^2 + 2\tfrac{\dot{\mathcal{F}}}{\mathcal{F}}\dot{\varphi}\varphi \Big]$$

$$+ k(d-3)\mathcal{F}\Omega\varphi^{(d-4)} + \tfrac{2}{(d-2)}\mathcal{F}^{-1}\Omega^{-1}\varphi^{(d-2)}\dot{\chi}^2 \Big\}. \tag{4}$$

The variation of S respect to $\mathcal{F}$, Ω and φ yields

$$2\frac{\ddot{\varphi}}{\varphi} - 2\frac{\dot{\varphi}\dot{\Omega}}{\varphi\Omega} + (d-3)\frac{\dot{\varphi}^2}{\varphi^2} - k(d-3)\Omega^2\varphi^{-2} + \frac{2}{(d-2)}\mathcal{F}^{-2}\dot{\chi}^2 = 0, \tag{5}$$

$$(d-3)\frac{\dot{\varphi}^2}{\varphi^2} + 2\frac{\dot{\mathcal{F}}}{\mathcal{F}}\frac{\dot{\varphi}}{\varphi} - k(d-3)\Omega^2\varphi^{-2} + \frac{2}{(d-2)}\mathcal{F}^{-2}\dot{\chi}^2 = 0 \tag{6}$$

and

$$\frac{(d-4)(d-3)}{2}\left(\frac{\dot{\varphi}^2}{\varphi^2} - k\Omega^2\varphi^{-2}\right) + (d-3)\left[\frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{\mathcal{F}}}{\mathcal{F}} - \frac{\dot{\Omega}}{\Omega}\right) + \frac{\ddot{\varphi}}{\varphi}\right] - \frac{\dot{\mathcal{F}}\dot{\Omega}}{\mathcal{F}\Omega} + \frac{\ddot{\mathcal{F}}}{\mathcal{F}} - \mathcal{F}^{-2}\dot{\chi}^2 = 0, \tag{7}$$

respectively, while the variation respect to χ gives

$$\frac{d}{dr}\{\mathcal{F}^{-1}\Omega^{-1}\varphi^{(d-2)}\dot{\chi}\} = 0. \tag{8}$$

Now, by combining (6) and (5) we get the relation

$$\frac{d}{dr}\left\{\ln(\mathcal{F}^{-1}\Omega^{-1}\dot{\varphi})\right\} = 0, \tag{9}$$

We observe that (8) and (9) imply

$$\mathcal{F}^{-1}\Omega^{-1}\varphi^{(d-2)}\dot{\chi} = A \tag{10}$$

and

$$\mathcal{F}^{-1}\Omega^{-1}\dot{\varphi} = B, \tag{11}$$

respectively, where A and B are integration constants. Thus, from (10) and (11) we obtain a relation between the electric radial field $\dot{\chi}$ and the function φ, namely

$$\dot{\chi} = C\frac{\dot{\varphi}}{\varphi^{(d-2)}}, \tag{12}$$

where $C = \frac{A}{B}$.

Of course, these results can be obtained if one starts with Einstein-Maxwell field equations. In fact, the Einstein field equations can be obtained by making variations of the action (1) respect to the metric γ, resulting

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R = 2T_{\mu\nu}, \tag{13}$$

where the energy momentum tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}\gamma_{\mu\nu}F_{\lambda\tau}F^{\lambda\tau}. \tag{14}$$

Considering the metric (2) we note that the first two diagonal elements of the energy-momentum tensor are given by $T_{00} = \frac{1}{2}\Omega^{-2}\dot{\chi}^2$ and $T_{11} = -\frac{1}{2}\mathcal{F}^{-2}\dot{\chi}^2$. Taking into account these expressions for T_{00} and T_{11}, and adding components G_{00} and G_{11} of the field equations in (13), we see that the following relation holds:

$$\Omega^2 R_{00} + \mathcal{F}^2 R_{11} = 0. \tag{15}$$

If we substitute R_{00} and R_{11} from the Appendix, we obtain again (9).

By making variations of (1) with respect the gauge field A_μ the Maxwell field equation can also be obtained;

$$\nabla_\mu F^{\mu\nu} = 0, \tag{16}$$

where ∇_μ stands for covariant derivative. By taking the zero component of the Maxwell equation (16) we obtain basically (8) after a direct computation, and this in turn implies the solution (10).

3. Charged black holes and Lagrangian formalism

Variational methods for the case of black holes have been used extensively in the past [15]-[18]. Now we present a direct method that takes the radial parameter r as the fundamental independent variable. Getting back to the initial action, we remark that the integrand in (4) can be interpreted as a type of Lagragian $\mathcal{L}$. We introduce the 'coordinates' $q^a \in \{q^1, q^2, q^3\}$ given by $\varphi = e^{q^1}$, $\mathcal{F} = e^{q^2}$, $\chi = q^3$, in such a way that $\mathcal{L}$ can be written as

$$\mathcal{L} = \frac{1}{2}\lambda^{-1}\left[(d-3)\left(\dot{q}^1\right)^2 + 2\dot{q}^1\dot{q}^2 + \frac{2}{(d-2)}e^{-2q^2}\left(\dot{q}^3\right)^2\right] + \frac{1}{2}\lambda m_o^2, \tag{17}$$

where

$$\lambda = \varphi^{-(d-2)}\mathcal{F}^{-1}\Omega, \tag{18}$$

which plays the role of a Lagrange multiplier, and

$$m_o^2 = k(d-3)\mathcal{F}^2 \varphi^{2(d-3)} \tag{19}$$

is the analogue of a mass term.

The Euler-Lagrange equations obtained from (17), by making variations of q^1, q^2 and q^3 respectively, become

$$\frac{d}{dr}\left\{\lambda^{-1}\left[(d-3)\dot{q}^1 + \dot{q}^2\right]\right\} - \lambda(d-3)m_o^2 = 0, \tag{20}$$

$$\frac{d}{dr}\left\{\lambda^{-1}\dot{q}^1\right\} + \lambda^{-1}\frac{2}{(d-2)}e^{-2q^2}\left(\dot{q}^3\right)^2 - \lambda m_o^2 = 0, \tag{21}$$

and

$$\frac{d}{dr}\left[\lambda^{-1}e^{-2q^2}\left(\dot{q}^3\right)\right] = 0. \tag{22}$$

The variation respect to λ gives

$$\lambda^{-2}\left[(d-3)\left(\dot{q}^1\right)^2 + 2\dot{q}^1\dot{q}^2 + \frac{2}{(d-2)}e^{-2q^2}\left(\dot{q}^3\right)^2\right] - m_o^2 = 0. \tag{23}$$

Before we continue, it is worthwhile mentioning that the equations (20)-(23) are consistent with equations (5)-(8). In fact, using the definition of the coordinates $\{q^a\}$ and performing the corresponding derivatives, we note first that (22) is just (8), and also that, up to a multiplicative factor, (23) is equivalent to (6). Now, multiplying (21) by $(d-3)$ and substracting the result to (20), we obtain

$$\frac{d}{dr}\left\{\lambda^{-1}\dot{q}^2\right\} - 2\lambda^{-1}\left(\frac{d-3}{d-2}\right)e^{-2q^2}\left(\dot{q}^3\right)^2 = 0. \tag{24}$$

After some computation it can be shown that (7) follows from (21), (23) and (24). Finally, it is straightforward to get the equation (5) from (21) and (23).

Equation (22) can be solved and gives

$$\lambda^{-1}e^{-2q^2}\dot{q}^3 = A, \tag{25}$$

which is equivalent to (10). Using (24) and (25) one gets

$$\frac{d}{dr}\left\{\lambda^{-1}\dot{q}^2\right\} - 2A\left(\frac{d-3}{d-2}\right)\dot{q}^3 = 0, \tag{26}$$

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This relation will be useful below. Now, equations (20) and (23) imply that

$$\frac{d}{dr}\left\{\lambda^{-1}\left[(d-3)\dot{q}^1 + \dot{q}^2\right]\right\} = (d-3)\lambda^{-1}\left[(d-3)\left(\dot{q}^1\right)^2 + 2\dot{q}^1\dot{q}^2 + \frac{2}{(d-2)}e^{-2q^2}\left(\dot{q}^3\right)^2\right].$$
(27)

Making use of (24) in (27) we get, after simplifications,

$$\frac{d}{dr}\left[\ln\left(\lambda^{-1}\dot{q}^1\right) - (d-3)q^1 - 2q^2\right] = 0.$$
(28)

Considering the definitions of (q^1, q^2, q^3) in terms of $(\varphi, \mathcal{F}, \chi)$, equations (26) and (28) leads to

$$\lambda^{-1}\frac{\dot{\mathcal{F}}}{\mathcal{F}} - 2A\left(\frac{d-3}{d-2}\right)\chi = D$$
(29)

and

$$\lambda^{-1}\dot{\varphi}\varphi^{-(d-2)}\mathcal{F}^{-2} = B,$$
(30)

respectively. Here, D and B are integration constants in agreement with (11). Relations (25) and (30) provide a solution for the electric field $\dot{\chi}$ in terms of $\dot{\varphi}$ and φ:

$$\dot{\chi} = \frac{A}{B}\frac{\dot{\varphi}}{\varphi^{(d-2)}},$$
(31)

which is just (12). By combining (30) with this equation we obtain that $\lambda^{-1} = A\frac{\mathcal{F}^2}{\dot{\chi}}$, which can be inserted into (29), resulting in

$$A\mathcal{F}\dot{\mathcal{F}} - 2A\left(\frac{d-3}{d-2}\right)\chi\dot{\chi} = D\dot{\chi}.$$
(32)

One can express (32) as

$$\frac{d}{dr}[A\mathcal{F}^2 - 2D\chi - 2A\left(\frac{d-3}{d-2}\right)\chi^2] = 0.$$
(33)

Now, from (31) one notes that

$$\chi = \frac{-C}{(d-3)\varphi^{(d-3)}} + G,$$
(34)

where G is integration constant. Thus, by substituting (34) into (33) one obtains

$$\mathcal{F}^2 = \left[\frac{E}{A} + \frac{2DG}{A} + 2\left(\frac{d-3}{d-2}\right)G^2\right]$$

$$-\left(\frac{2D}{B(d-3)} + \frac{4CG}{(d-2)}\right)\frac{1}{\varphi^{(d-3)}} + \frac{2C^2}{(d-2)(d-3)\varphi^{2(d-3)}}.$$
(35)

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Here, the quantity E is another integration constant.

Recalling that $\mathcal{F} = e^{\frac{f}{2}}$, then the γ_{00} component of the metric $\gamma_{\mu\nu}$ can be written as

$$\gamma_{00} = -e^f = -\left(K - \frac{2\mu}{\varphi^{(d-3)}} + \frac{\vartheta^2}{\varphi^{2(d-3)}} \right), \tag{36}$$

where we used the definitions $K = \left(\frac{E}{A} + \frac{2DG}{A} + 2\left(\frac{d-3}{d-2}\right)G^2 \right)$, $\mu = \frac{D}{B(d-3)} + \frac{2CG}{(d-2)}$ and $\vartheta^2 = \frac{2C^2}{(d-2)(d-3)}$.

Since from (11) $\Omega = B^{-1}\dot{\varphi}\mathcal{F}^{-1}$ [see also eq. (30)], we have that

$$\gamma_{11} = e^h = B^{-1}\dot{\varphi}e^{-f}. \tag{37}$$

4. Charged black holes and Hamiltonian formalism

Now we turn our attention to a slightly different view of our initial problem that resembles the analysis for a relativistic point particle. This approach has been proved to be fruitful in several ways, specially for quantization via the Dirac constrained systems formalism. The Lagrangian (17) can be written in a more concise form

$$\mathcal{L} = \frac{1}{2}\left\{ \lambda^{-1}\dot{q}^a\dot{q}^b\xi_{ab} + \lambda m_o^2 \right\}. \tag{38}$$

Here, we have defined the metric ξ_{ab} with components

$$[\xi_{ab}] = \begin{pmatrix} (d-3) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{2}{(d-2)}e^{-2q^2} \end{pmatrix}. \tag{39}$$

The inverse metric corresponding to (39) is

$$[\xi^{ab}] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -(d-3) & 0 \\ 0 & 0 & \frac{1}{2}(d-2)e^{2q^2} \end{pmatrix}. \tag{40}$$

Diagonalizing (42), we see that it has three eigenvalues, given by

$$\phi_1 = \frac{(d-3)+\sqrt{(d-3)^2+4}}{2},$$

$$\phi_2 = \frac{(d-3)-\sqrt{(d-3)^2+4}}{2}, \tag{41}$$

$$\phi_3 = \frac{2}{(d-2)}e^{-2q^2}.$$

We note that in $d = 4$, ϕ_1 is precisely the golden ratio [3]. Furthermore, in order to be consistent with the corresponding eigenvectors, it is useful to define the new coordinates

$$w^1 = \frac{1}{\sqrt{\phi_1 - \phi_2}} \left(\phi_2 q^1 + q^2 \right),$$

$$w^2 = \frac{1}{\sqrt{\phi_1 - \phi_2}} \left(\phi_1 q^1 + q^2 \right), \tag{42}$$

$$w^3 = q^3.$$

In terms of these variables, (38) can be expressed as follows:

$$\mathcal{L} = \frac{1}{2} \left\{ \lambda^{-1} \dot{w}^a \dot{w}^b \eta_{ab} + \lambda m_o^2 \right\}. \tag{43}$$

Although we have a similar form between (38) and (43), we have that now the associated metric has the diagonal form $\eta_{ab} = diag\left(-1, 1, \frac{2}{(d-2)} e^{-2q^2} \right)$.

Let us make some brief comments about the gauge fixing associated with (38). As it has been just shown, the Lagrangian (38) has been expressed in a similar form as the relativistic point particle, whose Lagrangian L_{rel} goes as

$$L_{rel} = \frac{1}{2} \Lambda^{-1} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu} - \frac{1}{2} \Lambda M_0^2, \tag{44}$$

where Λ is a Lagrange multiplier, M_0 is the mass of the system, $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$ and τ is an arbitrary parameter. Varing (44) with respect to λ leads to

$$\Lambda^{-2} \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} + M_0^2 = 0, \tag{45}$$

We observe that in this case (see Refs [13] and [14] for details), since M_0 is a constant, it is useful to make the choice $\Lambda = \frac{1}{M_0}$. We get

$$\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = -1, \tag{46}$$

or

$$dx^\mu dx_\mu = -d\tau^2. \tag{47}$$

In general we have the invariance of the line element

$$dx^\mu dx_\mu = dx'^\mu dx'_\mu. \tag{48}$$

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So, (47) can be obtained from (48) by setting $x'^0 = \tau$ and $x'^i = 0$. This prove that the choice $\Lambda = \frac{1}{M_0}$ is equivalent to choose a reference system with $x'^i = 0$. It is worth mentioning that this means that τ must be identified with the proper time.

In our case, the expression (38) gives

$$\lambda^{-2}\dot{q}^a\dot{q}^b\xi_{ab} - m_o^2 = 0. \tag{49}$$

Comparing (45) and (49) we observe that both equations are very similar. However, while in (45) M_0^2 is a constant, one notes from (19) that m_0^2 is not. Moreover, while the $\eta_{\mu\nu}$ is diagonal ξ_{ab} is not. Hence we have more freedom to choose λ. A convenient choice is

$$\lambda = \frac{(d-3)^{1/2}e^{-q^1}}{m_0} \tag{50}$$

Thus, (49) leads to

$$dq^a dq^b \xi_{ab} = (d-3)e^{-2q^1}dr^2. \tag{51}$$

Since one should have the invariant

$$dq^a dq^b \xi_{ab} = dq'^a dq'^b \xi'_{ab}. \tag{52}$$

It is not difficult to see that (51) can be obtained from (52) by setting $dq'^2 = 0$, $dq'^3 = 0$ and taking $dq^1 = dq'^1 = e^{-q^1}dr$. Since $\varphi = e^{q^1}$, this is equivalent to choose a reference frame such that

$$d\varphi = dr. \tag{53}$$

Let us set $\varphi = r$. With this choice it is straightforward to see from (31) and (36) that both $dq'^2 = 0$ and $dq'^3 = 0$ can be considered as conditions for $r \to \infty$. The condition $\varphi = r$ is justified in the sense that it leads to the right Newtonian limit, when one assumes asymptotic flatness. In this limit, in (36) one can set $K = 1$ and [4][19]

$$\mu = \frac{4G_d\Gamma\left(\frac{d-1}{2}\right)}{(d-2)\pi^{(d-3)/2}}M. \tag{54}$$

Here M is the mass of the black-hole and $\Gamma(n)$ is the gamma function. Moreover, ϑ is related with the charge of the black hole Q (in Gaussian units) by mean of

$$\vartheta^2 = \frac{2G_d}{(d-2)(d-3)}Q^2. \tag{55}$$

To recover the usual Reissner-Nordström solution, we set $B = 1$ in (37). Of course, in the limit where $Q = 0$, the metric (2) corresponds to the usual Schwarzschild-Tangherlini metric [20].

By reviewing (36) and (37), we observe that two horizons arise, as is well known, given by

$$\varphi_{\pm} = \left(\mu \pm \sqrt{\mu^2 - \vartheta^2} \right)^{\frac{1}{(d-3)}}.$$

(56)

Until now, we have made the Lagrangian for a charged black hole in $d \geq 4$. We now turn our attention to the Hamiltonian analysis. The canonical moments of (q^1, q^2, q^3) corresponding to (17) are given by $p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}$. Explicitly:

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}^1} = \lambda^{-1} \left[(d-3)\dot{q}^1 + \dot{q}^2 \right],$$

$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}^2} = \lambda^{-1} \dot{q}^1,$$

(57)

$$p_3 = \frac{\partial \mathcal{L}}{\partial \dot{q}^3} = \frac{2}{(d-2)} \lambda^{-1} e^{-2q^2} \dot{q}^3.$$

By defining the Legendre transform of (38) as

$$\mathcal{H}_0 = \dot{q}^a p_a - \mathcal{L},$$

(58)

we obtain from (57) that

$$\mathcal{H}_0 = \frac{\lambda}{2} \left\{ \left[2p_1 p_2 - (d-3)(p_2)^2 + \frac{1}{2}(d-2)e^{2q^2}(p_3)^2 \right] - m_o^2 \right\}.$$

(59)

This can be shown to be equivalent to

$$\mathcal{H}_L = \mathcal{L} - \lambda m_o^2.$$

(60)

Also, the Lagrangian (38) can be written as

$$\mathcal{L}' = \dot{q}^a p_a - \frac{\lambda}{2} \left\{ \left[2p_1 p_2 - (d-3)(p_2)^2 + \frac{1}{2}(d-2)e^{2q^2}(p_3)^2 \right] - m_o^2 \right\},$$

(61)

or using (40), as

$$\mathcal{L}' = \dot{q}^a p_a - \frac{\lambda}{2} \left\{ p_a p_b \xi^{ab} - m_o^2 \right\}.$$

(62)

It is suggestive to call the expression inside the bracket $\mathcal{H}_L$, since the variation of (62) respect to λ yields

$$\mathcal{H}_L = p_a p_b \xi^{ab} - m_o^2 = 0.$$

(63)

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This constraint can be obtained by substituting (57) in $\mathcal{H}_L$, that yields the same relation (23).

From here, it would interesting to explore the implications of applying the Dirac's quantization formalism for constrained systems.

5. Final remarks

In this work we have obtained the known Reisner-Nordström black hole solution for arbitrary dimensions ($d \geq 4$). This solution was obtained by mean of a Lagrangian approach that had not been explored in the literature until very recently [3].

More precisely, starting with the most general metric associated to radial symmetry, we proceeded to consider the problem as an analogue to that of a relativistic point particle. For this purpose, a convenient choice of independent coordinates allowed us to take the analysis with a reduced metric. We argue that the transformations performed makes the new metric suitable for quantization according to the Dirac's constrained systems formalism. Although this idea is well known and form the basis for several approaches in quantum gravity [23]-[25], our method for obtaining the solutions, as well as the implications that it can have for quantization, do not seem to have been explored before.

We showed that our procedure applies for both Reisner-Nordström and Schwarzschild solutions in $d > 3$. The applicability of the method appears to be directly related to spherical symmetry, that in turns leads to treat the static models (i.e. Schwarzschild-Tangherlini and Reissner-Nordström) as constrained systems with only one first class constraint. Nevertheless, it is interesting to explore also the case of stationary solutions from this perspective. In this case, in order to obtain the Kerr-Newman or Myers-Perry metrics with our procedure, we expect that the introduction of new constraints permits to get rid of effects like frame dragging, simplifying thus the analysis [2][22].

Also, as we have seen, the initial problem was simplified to a reduced metric that can be diagonalized, in such a way that it can be suitable for analysing quantum aspects of black holes [23][24]. Both the procedure and the analysis used in this work can have implications to both cosmological models in higher dimensions and alternative theories of gravity (see for instance [26]-[29] as well as references therein). This type of explorations will be reported elsewhere.

Appendix. The Riemann and Ricci tensor.

From the metric (2), we find that the only nonvanishing Christoffel symbols, defined by $\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\lambda}(g_{\lambda\beta,\alpha} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda})$, are

$$\Gamma^0_{01} = \frac{1}{2}\dot{f} \qquad \Gamma^1_{00} = \frac{1}{2}\dot{f}e^{f-h} \qquad \Gamma^1_{11} = \frac{1}{2}\dot{h}$$

$$\Gamma^1_{ij} = -e^{-h}\varphi\dot{\varphi}\tilde{\gamma}_{ij} \qquad \Gamma^i_{1j} = \varphi^{-1}\dot{\varphi}\delta^i_j \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}.$$

(A.1)

The Riemann tensor is defined by $R^\alpha_{\ \mu\beta\nu} = \partial_\beta\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\beta\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}$, and we have as relevant components, according to (A.1):

$$R^0_{\ 101} = -\frac{1}{2}\ddot{f} + \frac{1}{4}\dot{f}\dot{h} - \frac{1}{4}\dot{f}^2, \qquad R^1_{\ 010} = -e^{f-h}\left(-\frac{1}{2}\ddot{f} + \frac{1}{4}\dot{f}\dot{h} - \frac{1}{4}\dot{f}^2\right),$$

$$R^0_{\ i0j} = -\frac{1}{2}e^{-h}\tilde{\gamma}_{ij}\varphi\dot{\varphi}\dot{f}, \qquad R^i_{\ 0j0} = \frac{1}{2}e^{f-h}\delta^i_j\varphi^{-1}\dot{\varphi}\dot{f},$$

$$R^1_{\ i1j} = e^{-h}\tilde{\gamma}_{ij}\left(\frac{1}{2}\varphi\dot{\varphi}\dot{h} - \varphi\ddot{\varphi}\right), \qquad R^i_{\ 1j1} = \delta^i_j\varphi^{-1}\left(\frac{1}{2}\dot{\varphi}\dot{h} - \ddot{\varphi}\right),$$

$$R^i_{\ jkl} = \tilde{R}^i_{\ jkl} + e^{-h}\dot{\varphi}^2\left(\delta^i_l\tilde{\gamma}_{jk} - \delta^i_k\tilde{\gamma}_{jl}\right).$$

(A.2)

Next, the Ricci tensor $R_{\mu\nu} = R^\alpha_{\ \mu\alpha\nu}$ has components

$$R_{00} = e^{f-h}\left(\frac{1}{2}\ddot{f} - \frac{1}{4}\dot{f}\dot{h} + \frac{1}{4}\dot{f}^2 + \frac{D-2}{2}\varphi^{-1}\dot{\varphi}\dot{f}\right),$$

$$R_{11} = -\frac{1}{2}\ddot{f} + \frac{1}{4}\dot{f}\dot{h} - \frac{1}{4}\dot{f}^2 + (D-2)\varphi^{-1}\left(\frac{1}{2}\dot{\varphi}\dot{h} - \ddot{\varphi}\right), \qquad (A.3)$$

$$R_{ij} = e^{-h}\tilde{\gamma}_{ij}\left[-\frac{1}{2}\varphi\dot{\varphi}\dot{f} + \frac{1}{2}\dot{h}\varphi\dot{\varphi} - \varphi\ddot{\varphi} - (d-3)\dot{\varphi}^2 + k(d-3)e^h\right].$$

We note that we have used the fact that the maximally symmetric subspace defined by the metric $\tilde{\gamma}_{ij}$ demands that $\tilde{R}_{ij} = \tilde{\gamma}^{kl}\tilde{R}_{kilj} = k(d-3)\tilde{\gamma}_{ij}$. This in turn implies that $\tilde{R} = k(d-2)(d-3)$, that is used to obtain the curvature tensor $R = g^{\mu\nu}R_{\mu\nu}$ as

$$R = e^{-h}(d-2)\left[\varphi^{-1}\dot{\varphi}(\dot{h} - \dot{f}) - 2\varphi^{-1}\ddot{\varphi} - (d-3)\varphi^{-2}\dot{\varphi}^2\right] +$$

(A.4)

$$+e^{-h}\left(-\ddot{f} + \frac{1}{2}\dot{f}\dot{h} - \frac{1}{2}\dot{f}^2\right) + k(d-2)(d-3)\varphi^{-2}.$$

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